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On the well-posedness of a two-phase minimization problem ☆

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ABSTRACT

We prove a series of results concerning the emptiness and non-emptiness of a certain set of Sobolev functions related to the well-posedness of a two-phase minimization problem, involving both the $p(x)$-norm and the infinity norm. The results, although interesting in their own right, hold the promise of a wider applicability since they can be relevant in the context of other problems where minimization of the p-energy in a part of the domain is coupled with the more local minimization of the L^∞-norm on another region.

1. Introduction

Let D and Ω, $D \subset \Omega$, be bounded and convex domains in $\mathbb{R}^N$, with C^1-smooth boundaries, and consider the elliptic problem

$$\begin{cases} -\Delta_{p(x)} u(x) = 0, & x \in \Omega, \\ u(x) = f(x), & x \in \partial\Omega, \end{cases} \tag{1}$$

where $\Delta_{p(x)}$ is formally defined by

$$\Delta_{p(x)} u = \begin{cases} \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) & \text{if } p(x) < +\infty, \\ \Delta_\infty u = \sum_{i,j=1}^N \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} & \text{if } p(x) = +\infty. \end{cases} \tag{2}$$

Here the boundary data f is Lipschitz and the variable exponent $p(x)$ is a continuously differentiable bounded function in $\overline{\Omega} \setminus D$ that satisfies the two conditions

$$p_- := \inf_{x\in\Omega} p(x) > N \tag{3}$$

and

$$p(x) = +\infty, \quad x \in D. \tag{4}$$

The problem was recently studied in [8], where the existence of a suitable solution is obtained, together with its characterization as the unique minimizer of the variational problem

$$\min_{u\in S} \int\limits_{\Omega\setminus\overline{D}} \frac{|\nabla u|^{p(x)}}{p(x)}\, dx$$

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in the admissible set

$$S=\left\{u\in W^{1,p_-}(\Omega)\colon\ u|_{\Omega\setminus\overline{D}}\in W^{1,p(x)}(\Omega\setminus\overline{D}),\ \|\nabla u\|_{L^\infty(D)}\leqslant 1 \text{ and } u|_{\partial\Omega}=f\right\},\tag{5}$$

which is, in addition, ∞-harmonic within D, i.e., a viscosity solution of the infinity Laplacian equation

$$-\Delta_\infty u=0.$$

This nonlinear and strongly degenerate elliptic PDE seems to be ubiquitous [7] and has recently been connected to yet another application, namely the probabilistic description of certain tug-of-war games [10]. Questions related to the behavior of p-harmonic functions in the limit as $p\to\infty$ have been widely studied after the pioneering findings of [3]. Variable exponents, in turn, appear in connection to important applications, namely in elasticity [12] and the modeling of electrorheological fluids [1].

A crucial role in [8], where the limit problem the solution solves (in the viscosity sense) is also identified, is played by the set S since the main results depend critically on whether it is non-empty or not. Thus, the well-posedness of this two-phase minimization problem is directly related to understanding when exactly that happens. It is obvious that the non-emptiness of S will depend on the geometry of the problem and on the boundary data f. For example, if $\partial\Omega\cap\overline{D}=\emptyset$, S is always non-empty, an element in it being very easy to obtain extending a constant function in D. When $\partial\Omega\cap\overline{D}\neq\emptyset$, the condition that the Lipschitz constant of $f|_{\partial\Omega\cap\overline{D}}$ is less than or equal to one is necessary but, in general, it is not sufficient for the non-emptiness of S (cf. Section 4 in [8], where it is explained by counter-example why the obvious approach does not necessarily work; our Theorem 4.1 provides explicit counter-examples). As it happens, what at first was overlooked as a straightforward matter became an interesting challenge.

In this note we shed further light into the problem, identifying conditions that guarantee the non-emptiness (and the emptiness) of the set S. A characterization, in its full generality, remains open. We feel the results hold the promise of a wider applicability although they are interesting in their own right. Indeed, similar questions are bound to arise in relation to other problems where minimization of the p-energy in a part of the domain is coupled with the more local minimization of the L^∞-norm on another region. Of particular interest seem to be certain relations, like (11) below, between the variable exponent $p(\cdot)$ and some geometric properties related to the way the boundaries of Ω and D interact.

The paper is organized as follows: Section 2 collects the notation used throughout the paper; Section 3 identifies conditions that guarantee the non-emptiness of S; Section 4 treats the emptiness case; finally, in Section 5, we present several examples illustrative of the main results. We suggest the reader starts out with reading the examples in the last section of the paper. Although that is where they belong in the context of a consistently written text, mainly for notational and definiteness reasons, it can be a significant help in the understanding of the proofs to have prior contact with concrete examples of the objects involved. Another source of simplification is to consider, on first reading, that the exponent $p(\cdot)$ is constant and sufficiently large; in fact, as is clear from the proofs, the general case is reduced essentially to this one. We opted to present the proofs in the general setting to avoid an unnecessary duplication of arguments.

2. Notation

In this section, we collect a set of notation that will be used in the sequel. Since the matter is quite technical, we thought it would help the reader to resort to this section whenever notational doubts arise.

For $N>1$, let $\mathbb{R}^N_+$ be the half-space consisting of the vectors with positive N-th coordinate and let $\mathbb{R}^{N-1}_*$ be its boundary. We also define the following sets (and assume all are non-empty to avoid trivialities):

$$\Omega_*=\Omega\setminus\overline{D},$$

$$Q:=\partial D\cap\partial\Omega,$$

$$Q_*=Q\cap\partial\Omega_*,$$

$$W_\varepsilon:=\bigcup_{x\in Q}B_\varepsilon(x)$$

where $B_\varepsilon(x)$ is the open ball of radius ε, centered at x.

For $z\in\mathbb{R}^N$, define the set

$$U_z(Q)=\left\{x\in\mathbb{R}^N\colon\ |x-z|\leqslant|x-y|,\ \forall y\in Q\right\}$$

of the points which are closer or at the same distance to z than to the points of Q.

For $z\in\mathbb{R}^N\setminus D$, let $d(z)$ be the point of ∂D which is closest to z; it is unique due to the convexity of D.

For $z\in Q_*$, we define

$$\begin{aligned}\gamma(z):=\sup\bigl\{&\gamma\in[0,1]\colon\ \exists C>0 \text{ and an open neighbourhood } W(z) \text{ of } z,\\ &|w-z|\leqslant C\left|w-d(w)\right|^\gamma,\ \forall w\in\partial\Omega\cap W(z)\cap U_z(Q)\bigr\}\end{aligned}\tag{6}$$

and

$$\overline{\gamma}(z) := \inf\big\{\gamma \in [0,1]\colon \exists C > 0 \text{ and an open neighbourhood } W(z) \text{ of } z,$$
$$|w - z| \geqslant C\big|w - d(w)\big|^{\gamma},\ \forall w \in \partial\Omega \cap W(z) \cap U_z(Q)\big\}. \tag{7}$$

Note that the supremum and infimum always exist since the inequalities in (6) and (7) are fulfilled, resp., for $\gamma = 0$ and $\gamma = 1$. The first is trivial, and the second follows from the C^1-smoothness of $\partial\Omega$ and ∂D.

Let f be a fixed Lipschitz scalar function defined on $\partial\Omega$. For any subset K of $\mathbb{R}^N$, let

$$L(K) := \min\big\{L\colon \big|f(x) - f(y)\big| \leqslant L|x - y|,\ \forall x, y \in K \cap \partial\Omega\big\}.$$

It is easy to see that the minimum always exists. We say that

(A) f *is of type* A if $L(W) \leqslant 1$ for some open neighborhood W of Q;
(B) f *is of type* B if $L(Q) \leqslant 1$, f is not of type A, and

$$\forall \varepsilon > 0, \exists \delta > 0\colon \big|f(x) - f(q)\big| \leqslant (1+\varepsilon)|x - q|, \quad \forall q \in Q, \forall x \in \partial\Omega \cap W_\delta;$$

(C) f *is of type* C if $L(Q) \leqslant 1$, and

$$\exists \varepsilon > 0\colon \forall \delta > 0, \exists q \in Q, \exists x \in \partial\Omega \cap W_\delta\colon \big|f(x) - f(q)\big| > (1+\varepsilon)|x - q|.$$

Finally, let S_L be the subset of S consisting of all functions which are Lipschitz in $\overline{\Omega}$.

The symbol c may stand for a generic positive constant that can take different values in different lines.

3. Non-emptiness of *S*

We start with a result that holds for D and Ω not necessarily of class C^1; see Example 5.4 for an insight.

Theorem 3.1. *Assume there are an open neighborhood W of Q and mutually disjoint sets W_1 and W_2 such that $W \cap \partial\Omega = W_1 \cup W_2$, $Q \subset W_1$, $L(W_1) \leqslant 1$ and*

$$\exists C > 0\colon \min_{z \in Q} |x - z| \leqslant C|x - y|, \quad \forall x \in W_2, \forall y \in \partial D \cap W.$$

Then S_L (and, consequently, S) is non-empty.

Proof. Let f_1 be a minimal Lipschitz extension [9,11,4] (see also [2]) of $f|_{W_1}$ to Ω. Then its Lipschitz constant (which coincides with the L_∞-norm of its gradient) does not exceed one. It suffices to prove that the function $f_2 : \partial\Omega \cup \overline{D} \to \mathbb{R}$ defined by

$$f_2(x) = \begin{cases} f(x) & \text{if } x \in \partial\Omega, \\ f_1(x) & \text{if } x \in \overline{D} \end{cases}$$

is Lipschitz, for in this case its Lipschitz extension to Ω is an element of S_L. Let $x \in \partial\Omega$, $y \in \overline{D}$. We have to check whether

$$\big|f_2(x) - f_2(y)\big| \leqslant c|x - y|, \tag{8}$$

for some constant c, independent of x, y. Without loss of generality, we assume $y \in \partial D$ (if not, we can replace it by the point of intersection of ∂D with the segment $[x, y]$) and $x, y \in W$ (a loss of the Lipschitz property of f_2 can only happen near Q). If $x \in W_1$, then (8) is clear (since f_2 coincides with f_1 in x and y, and f_1 is Lipschitz), whereas if $x \in W_2$, then for $z \in Q$ which has minimal (in Q) distance to x, one has

$$\begin{aligned}
\big|f_2(x) - f_2(y)\big| &\leqslant \big|f_2(x) - f_2(z)\big| + \big|f_2(z) - f_2(y)\big| \\
&= \big|f(x) - f(z)\big| + \big|f_1(z) - f_1(y)\big| \\
&\leqslant c\big(|x - z| + |z - y|\big) \\
&\leqslant c\big(2|x - z| + |x - y|\big) \\
&\leqslant (2C + 1)c|x - y|. \qquad \square
\end{aligned}$$

Remark 3.1. Observe that W_2 is always empty when D and Ω are of class C^1, so in this case $W_1 = W \cap \partial\Omega$. If D is not C^1-smooth, W_1 can even be Q (see Section 5).

We now identify several situations that guarantee that S is non-empty.

Theorem 3.2. (i) *If f is of type A, then $S_L \neq \emptyset$ (and, hence, $S \neq \emptyset$).*

(ii) *Let f be of type B. Assume that f may be decomposed as $f = f_A + f_0$, where f_A is of type A. Assume further that, for any $\varepsilon > 0$ and any $x \in (\partial\Omega \setminus Q) \cap W_\varepsilon$, there exist $z = z(x) \in Q_*$, $\gamma = \gamma(x) > 0$, and two constants $c_1, c_2 > 0$, that do not depend on x, such that*

$$|x - z| \leqslant c_1 \big|x - d(x)\big|^\gamma \tag{9}$$

and

$$\big|f_0(x)\big| \leqslant c_2 |x - z|^{\frac{1}{\gamma}}. \tag{10}$$

Then $S_L \neq \emptyset$.

(iii) *Let f be of type B or C and assume the set Q_* is finite. If*

$$p(z)\big(1 - \gamma(z)\big) < 1 + (N-1)\gamma(z), \quad \forall z \in Q_* \tag{11}$$

then $S \neq \emptyset$.

Remark 3.2. Since $p(z)$ is greater than N, condition (11) can be fulfilled only if $\gamma(z) > \frac{N-1}{2N-1}$.

Remark 3.3. The assumption that f is of type B in Theorem 3.2(ii) is made for the sake of presentation. As a matter of fact, the decomposition $f = f_A + f_0$ yields that f is either of type A or B, which can be checked directly, but also is a consequence of Theorem 4.1(i). Since the case of type A is already covered by Theorem 3.2(i), we keep only the case in which f is of type B.

Proof of Theorem 3.2. (i) This is a particular case of Theorem 3.1 with $W_1 = W \cap \partial\Omega$ and $W_2 = \emptyset$.

(ii) Due to part (i), we may assume that f_A is defined in $\overline{\Omega}$ and belongs to S_L. Consider the function $f_2 : \partial\Omega \cup \overline{D} \to \mathbb{R}$ defined by

$$f_2(x) = \begin{cases} f(x) & \text{if } x \in \partial\Omega, \\ f_A(x) & \text{if } x \in \overline{D}. \end{cases}$$

As in the proof of Theorem 3.1, it suffices to prove that this function is Lipschitz; moreover, it is enough to check (8) for $x \in \partial\Omega \cap W_\varepsilon$ and $y \in \partial D \cap W_\varepsilon$. If x is from Q, then (8) holds trivially, for f_A is Lipschitz. If not, then

$$\begin{aligned} \big|f_2(x) - f_2(y)\big| &= \big|f(x) - f_A(y)\big| \\ &\leqslant \big|f_0(x)\big| + \big|f_A(x) - f_A(y)\big| \\ &\leqslant c\big|x - d(x)\big| + c|x - y| \\ &\leqslant c|x - y|, \end{aligned}$$

due to (9) and (10).

(iii) Let f_1 be a minimal Lipschitz extension of $f|_Q$ to Ω. Then, in particular, $\|\nabla f_1\|_{L_\infty(D)} \leqslant 1$.

We will construct a function f_2 which is defined on a small closed neighbourhood Z of Q, coincides with f_1 and f, respectively, on $\overline{D} \cap Z$ and $\partial\Omega \cap Z$, and belongs to $W^{1,p(x)}(Z \cap \Omega_*)$ (see [5] for the definition of the variable exponent Sobolev spaces).

For any $z \in Q_*$, denote $Y_z = \Omega_* \cap Z_z$, where Z_z is a fixed and sufficiently small neighbourhood of z. Let H be the tangent hyperspace to $\partial\Omega$ at the point z. For any $s \in Y_z$, let $l(s)$ be the point of H which is closest to s (obviously unique), and let $m(s)$ and $k(s)$ be, respectively, the intersections of ∂D and $\partial\Omega$ with the straight line $(s, l(s))$. Finally, we define

$$f_2(s) = \frac{|s - k(s)| f_1(m(s)) + |s - m(s)| f(k(s))}{|k(s) - m(s)|} \tag{12}$$

as a convex combination.

The key point of the proof is to check if $f_2 \in W^{1,p(x)}(Y_z)$. We may suppose, w.l.o.g., that $z = 0$, $H = \mathbb{R}^{N-1}_*$ and $\Omega \subset \mathbb{R}^N_+$. Note that if we add the same Lipschitz function to f_1 and f, then, by (12), it would also be added to f_2, and this does not change the $W^{1,p(x)}$-regularity of f_2. Therefore we can also assume that $f_1 \equiv 0$ (if not, we may subtract f_1 from f and f_1).

Assume first that $N = 2$. Each point of $\mathbb{R}^2$ can be considered as a vector (x, y). The sets ∂D and $\partial\Omega$ are (locally, near z) graphs of C^1-smooth functions $\phi(x)$ and $\psi(x)$, respectively, with $\phi'(0) = \psi'(0) = 0$ and $\phi(x) \geqslant \psi(x)$. Then we rewrite (12) as

$$f_2(x, y) = \frac{(\phi(x) - y) f(k(x, y))}{\phi(x) - \psi(x)}, \quad (x, y) \in Y_z. \tag{13}$$

Observe that $k(x, y) = (x, \psi(x))$ does not depend on y and is a C^1-function of x. Since f is Lipschitz, we also have

$$\big|f_2(x, y)\big| \leqslant \big|f\big(k(x, y)\big)\big| \leqslant c\big|k(x, y)\big| \leqslant c|x|, \qquad \big|f'_x\big(k(x, y)\big)\big| \leqslant c. \tag{14}$$

Since $z \in \partial\Omega_*$, the set $Q \cap Z_z$ is either a graph of ψ, on an interval of the form $[0,a)$ or $(-a,0]$ $(a>0)$, or coincides with $\{z\} = \{(0,0)\}$. In the first two cases, both Y_z and $U_z(Q)$ lie, respectively, in the left or right half-planes. Moreover, $\overline{Y}_z \subset U_z(Q)$. In each of these cases, for any $\gamma < \gamma(z)$, by (6) with $w = k(x,y)$, we have

$$\begin{aligned}\phi(x) - \psi(x) &= \left|k(x,y) - m(x,y)\right| \geqslant \left|k(x,y) - d\big(k(x,y)\big)\right| \\ &\geqslant c\left|k(x,y)\right|^{\frac{1}{\gamma}} \geqslant c|x|^{\frac{1}{\gamma}}, \quad (x,y) \in Y_z. \end{aligned} \tag{15}$$

Fix some such γ sufficiently close to $\gamma(z)$. By (11), there exists $p_1 > p(z)$ such that

$$p_1(1-\gamma) < 1+\gamma. \tag{16}$$

Passing to a smaller neighbourhood Z_z if necessary, we may assume that $p(x,y) < p_1$. Thus, it suffices to prove $f_2 \in W^{1,p_1}(Y_z)$.

Taking into account (13) and (14), we get the bounds

$$\left|\frac{\partial f_2}{\partial x}(x,y)\right| \leqslant c\frac{|x|}{\phi(x)-\psi(x)} \tag{17}$$

and

$$\left|\frac{\partial f_2}{\partial y}(x,y)\right| \leqslant c\frac{|x|}{\phi(x)-\psi(x)}. \tag{18}$$

Thus, $f_2 \in W^{1,p_1}(Y_z)$ provided

$$\int_{-a}^{a}\int_{\psi(x)}^{\phi(x)} \left[\frac{|x|}{\phi(x)-\psi(x)}\right]^{p_1} dy\,dx < \infty, \tag{19}$$

for some small $a>0$ (here we assume for definiteness that $Q \cap Z_z = \{z\}$; the other cases can be treated analogously). But (19) holds since, due to (15),

$$\int_{\psi(x)}^{\phi(x)} \left[\frac{|x|}{\phi(x)-\psi(x)}\right]^{p_1} dy = \frac{|x|^{p_1}}{(\phi(x)-\psi(x))^{p_1-1}} \leqslant c|x|^{p_1+\frac{1}{\gamma}-\frac{p_1}{\gamma}}$$

and the exponent $p_1 + \frac{1}{\gamma} - \frac{p_1}{\gamma}$ is greater than -1 due to (16).

Let now $N>2$. Each element of $\mathbb{R}^N$ may be identified with a vector (x,y), where $x \in \mathbb{R}_*^{N-1}$ and $y \in \mathbb{R}$. The sets ∂D and $\partial\Omega$ near z are now graphs of C^1-smooth functions $\phi(x)$ and $\psi(x)$ defined for small $x \in \mathbb{R}^{N-1}$. The proof of the statement that $f_2 \in W^{1,p_1}(Y_z)$ is, essentially, analogous to the case $N=2$. Let us briefly describe the differences with respect to the two-dimensional proof. For $N>2$, $\partial\Omega$ is a manifold of dimension greater than one. Therefore Q is finite (if it is infinite, its boundary in the topology of $\partial\Omega$ is infinite as well, but this boundary should belong to Q_*, which is finite). Then the set $Q \cap Z_z$ coincides with $\{z\}$, and $\overline{Y}_z \subset U_z(Q)$, so (6) again implies (15). Furthermore, (16) becomes

$$p_1(1-\gamma) < 1+(N-1)\gamma. \tag{20}$$

Finally, (19) is replaced with

$$\int_O\int_{\psi(x)}^{\phi(x)} \left[\frac{|x|}{\phi(x)-\psi(x)}\right]^{p_1} dy\,dx \leqslant c\int_O |x|^{p_1+\frac{1}{\gamma}-\frac{p_1}{\gamma}}\,dx < \infty,$$

where O is a small neighbourhood of the origin in $\mathbb{R}_*^{N-1}$, and this holds since

$$p_1 + \frac{1}{\gamma} - \frac{p_1}{\gamma} > 1-N$$

due to (20).

For every $z \in Q \setminus Q_*$, define Z_z as some small neighbourhood of z which has no intersection with Ω_* and let $Y_z = \emptyset$ (remember that the sets Z_z and Y_z are already defined for $z \in Q_*$). Now the sets Z_z, $z \in Q$, form an open covering of Q. Due to the compactness of Q, it can be covered by a finite number of sets Z_z, and, w.l.o.g., Z belongs to the union of this finite number of Z_z. Then $Z \cap \Omega_*$ belongs to the union of the corresponding sets Y_z. Thus, $f_2 \in W^{1,p(x)}(Z \cap \Omega_*)$.

Define the function $f_3 : Z \cup \overline{D} \cup \partial\Omega \to \mathbb{R}$ as f_2 on Z, f_1 on $\overline{D}$ and f on $\partial\Omega$. The function f_3 is Lipschitz on $(Z \cup \overline{D} \cup \partial\Omega)\setminus W_Q$, where $W_Q \subset Z$ is a small neighbourhood of Q_*. Indeed, due to the compactness, it suffices to prove that the Lipschitz property holds locally. Then the loss of this property can happen only on $\overline{\Omega}_*$, but this is not the case since construction (12) is locally Lipschitz apart from Q_* (this becomes more evident using representation (13)), and we are separated from Q_* after cutting out W_Q. Now, we can continue f_3 from its range of definition to a function f_4 defined on $\overline{\Omega}$ and which is Lipschitz on $\overline{\Omega}\backslash W_Q$. Obviously, $f_4 \in W^{1,p(x)}(W_Q \cap \Omega_*) \subset W^{1,p_-}(W_Q \cap \Omega_*)$. But, since the function f_4 is continuous in $\overline{\Omega}$, and belongs to $W^{1,p_-}(W_Q \cap \Omega_*)$ and $W^{1,\infty}(W_Q \cap D)$, one has $f_4 \in W^{1,p_-}(W_Q \cap \Omega)$. Thus, $f_4 \in S$. $\square$

4. Emptiness of S

This section deals with sufficient conditions for S to be empty. We stress that none of these conditions is the negation of the conditions of the previous section so finding a necessary and sufficient condition for the non-emptiness of S is still an open problem.

Theorem 4.1. (i) *If f is of type C, then $S_L = \emptyset$.*

(ii) *Let the set Q_* be finite. If*

$$p(z)\big(1-\overline{\gamma}(z)\big) > 1 + (N-1)\overline{\gamma}(z), \tag{21}$$

for some $z \in Q_*$, then there exists a function f of type C so that $S = \emptyset$.

(iii) *If*

$$p(z)\big(1-\gamma(z)\big) > N, \tag{22}$$

for some $z \in Q_*$, then there exists a function f of type C so that $S = \emptyset$.

(iv) *If f is not of types A, B and C, then $S = \emptyset$.*

Remark 4.1. Since $p(z)$ is greater than N, condition (21) always holds provided $\overline{\gamma}(z) \leqslant \frac{N-1}{2N-1}$.

Proof of Theorem 4.1. (i) Let $\varepsilon > 0$ be such that for all $\delta > 0$ there exist $q_\delta \in Q$ and $x_\delta \in \partial\Omega \cap W_\delta$, with

$$\big|f(x_\delta) - f(q_\delta)\big| > (1+\varepsilon)|x_\delta - q_\delta|.$$

Let $y_\delta = d(x_\delta)$ and s_δ be any of the points of Q which are closest to x_δ. Simple geometrical analysis of the triangle $q_\delta x_\delta y_\delta$ shows that $|y_\delta - q_\delta| \leqslant |x_\delta - q_\delta|$ (since the largest side of any triangle opposes the largest angle). Further analysis of the geometry yields that the value of the angle $x_\delta s_\delta y_\delta$ vanishes as $\delta \to 0$, and thus

$$\lim_{\delta\to 0} \frac{|x_\delta - y_\delta|}{|x_\delta - s_\delta|} = 0.$$

If there exists an element $f_1 \in S_L$, then f_1 is Lipschitz, its Lipschitz constant is not greater than one on ∂D and it coincides with f on $\partial\Omega$. But, on the one hand,

$$\frac{|f_1(x_\delta) - f_1(y_\delta)|}{|x_\delta - q_\delta|} \leqslant c\frac{|x_\delta - y_\delta|}{|x_\delta - s_\delta|} \to 0 \quad \text{as } \delta \to 0,$$

and, on the other hand,

$$\begin{aligned}\frac{|f_1(x_\delta) - f_1(y_\delta)|}{|x_\delta - q_\delta|} &\geqslant \frac{|f_1(x_\delta) - f_1(q_\delta)|}{|x_\delta - q_\delta|} - \frac{|f_1(y_\delta) - f_1(q_\delta)|}{|x_\delta - q_\delta|} \\ &\geqslant (1+\varepsilon) - \frac{|y_\delta - q_\delta|}{|x_\delta - q_\delta|} \geqslant \varepsilon,\end{aligned}$$

so we arrive at a contradiction.

(ii) Let $Y_z = \Omega_* \cap Z_z$, where Z_z is a fixed and sufficiently small neighbourhood of the point z for which (21) holds. Let

$$f(w) = 2\min_{v\in Q}|w - v|, \quad w \in \partial\Omega. \tag{23}$$

It suffices to prove that, for any $f_0 \in S$, $f_0 \notin W^{1,p(x)}(Y_z)$. Let f_1 be a minimal Lipschitz extension of $f_0|_{\overline{D}}$ to Ω. Then it is enough to check that $f_2 = f_0 - f_1 \notin W^{1,p(x)}(Y_z)$. Note that $f_2 \equiv 0$ on $\overline{D}$ and

$$\begin{aligned}\big|f_2(w)\big| &\geqslant \big|f(w)\big| - \big|f_1(w) - f_1(z)\big| \\ &\geqslant 2|w-z| - |w-z| \\ &= |w-z|, \quad w \in \partial\Omega \cap U_z(Q).\end{aligned} \tag{24}$$

The proof follows closely that of Theorem 3.2(iii) and the notation is the same. Without loss of generality, we can suppose that $z = 0$, $H = \mathbb{R}_*^{N-1}$ and $\Omega \subset \mathbb{R}_+^N$. Let $N = 2$ (the generalization to higher dimensions is again straightforward). We are given the functions ϕ and ψ and coordinates (x, y). Putting $w = k(x, y)$ in (7), we get

$$\begin{aligned}\phi(x)-\psi(x) &= \bigl|k(x,y)-m(x,y)\bigr| \\ &\leqslant c\bigl|k(x,y)-d\bigl(k(x,y)\bigr)\bigr| \\ &\leqslant c\bigl|k(x,y)\bigr|^{\frac{1}{\gamma}} \\ &\leqslant c|x|^{\frac{1}{\gamma}}, \quad (x,y)\in Y_z,\end{aligned}$$

for any $\gamma > \overline{\gamma}(z)$. Observe that, by the sine law, the first inequality in this chain is a consequence of the fact that the angle

$$\bigl(k(x,y), m(x,y), d\bigl(k(x,y)\bigr)\bigr)$$

is separated from zero, which follows from the C^1-smoothness of D. Fix some such γ sufficiently close to $\overline{\gamma}(z)$. By (21), there exists $p_1 < p(z)$ such that

$$p_1(1-\gamma) > 1+\gamma. \tag{25}$$

Without loss of generality, $p(x,y) > p_1$. Thus, it suffices to prove that $f_2 \notin W^{1,p_1}(Y_z)$.

Assume the contrary. Then, in particular, for some small $a > 0$,

$$\int_{-a}^{a}\int_{\psi(x)}^{\phi(x)}\left|\frac{\partial f_2}{\partial y}(x,y)\right|^{p_1} dy\,dx \leqslant \infty.$$

As before, we assume for definiteness that $Q \cap Z_z = \{z\}$. A variational argument shows that the minimum of the functional

$$\int_{-a}^{a}\int_{\psi(x)}^{\phi(x)}\left|\frac{\partial g}{\partial y}(x,y)\right|^{p_1} dy\,dx \tag{26}$$

for

$$g \in W^{1,p_1}(Y_z), \qquad g|_{\partial\Omega\cup\partial D} = f_2|_{\partial\Omega\cup\partial D} \tag{27}$$

is achieved at a g that solves the PDE

$$\frac{\partial}{\partial y}\left(\left|\frac{\partial g}{\partial y}\right|^{p_1-2}\frac{\partial g}{\partial y}\right) = 0.$$

Note that in the variational procedure one has to integrate by parts only with respect to y, therefore it is possible to proceed without any boundary conditions on the "lateral" part of the boundary, i.e. at $x = \pm a$.

Clearly, such a g should be linear in y, so

$$g(x,y) = \frac{(\phi(x)-y)f_2(k(x,y))}{\phi(x)-\psi(x)}$$

due to (27). But this g cannot minimize (26). Indeed, due to (24),

$$\begin{aligned}\int_{\psi(x)}^{\phi(x)}\left|\frac{\partial g}{\partial y}(x,y)\right|^{p_1} dy &\geqslant \int_{\psi(x)}^{\phi(x)}\left[\frac{|k(x,y)|}{\phi(x)-\psi(x)}\right]^{p_1} dy \\ &\geqslant \frac{|x|^{p_1}}{(\phi(x)-\psi(x))^{p_1-1}} \\ &\geqslant c|x|^{p_1+\frac{1}{\gamma}-\frac{p_1}{\gamma}};\end{aligned}$$

as the exponent $p_1 + \frac{1}{\gamma} - \frac{p_1}{\gamma}$ is less than -1 by (25), the value of functional (26) on g is infinite.

(iii) Let $Y_z = \Omega_* \cap Z_z$, where Z_z is a fixed and sufficiently small neighbourhood of the point z for which (22) holds. Take f in the form (23), and assume that there exists $f_0 \in S$. Let f_1 be a minimal Lipschitz extension of $f_0|_{\overline{D}}$ to Ω, and $f_2 = f_0 - f_1$. As above, $f_2 \equiv 0$ on $\overline{D}$ and (24) holds.

There exists a neighbourhood of z, $C_z \subset \mathbb{R}^N$, with a C^1-smooth boundary, such that

$$Z_z \cap \Omega \subset C_z \subset \Omega.$$

Fix some $\gamma > \gamma(z)$ sufficiently close to $\gamma(z)$. By (22), there exists $p_1 < p(z)$ such that

$$p_1(1-\gamma) > N.$$

We may suppose that $p(x) > p_1$, $x \in C_z$. Then, $f_2 \in W^{1,p_1}(C_z)$. By Sobolev embedding, f_2 belongs to the Hölder class $C^\beta(\overline{C_z})$, with $\beta = 1 - \frac{N}{p_1} > \gamma$.

For any small neighbourhood $W(z)$ of z there exists $w \in \partial\Omega \cap W(z) \cap U_z(Q)$ such that

$$|w - z| > \left|w - d(w)\right|^\gamma.$$

By (24),

$$\left|f_2(w)\right| > \left|w - d(w)\right|^\gamma.$$

On the other hand,

$$\left|f_2(w)\right| = \left|f_2(w) - f_2\bigl(d(w)\bigr)\right| \leqslant c\left|w - d(w)\right|^\beta,$$

and thus

$$\left|w - d(w)\right|^{\gamma-\beta} \leqslant c;$$

so the distance between w and $d(w)$ is bounded from below, which contradicts the fact that $W(z)$ can be arbitrarily small.

(iv) If, on the contrary, $S \neq \emptyset$, then $L(Q) \leqslant 1$ (cf. [8]). Therefore, f has to be of one of the types A, B or C. $\square$

5. Examples

We finally gather a few examples that illustrate the findings of the previous sections and shed further light into its intricate reasonings.

Example 5.1. Let Ω be the unit disc $\{(x, y) \in \mathbb{R}^2\colon\ x^2 + y^2 < 1\}$, and let

$$D = \left\{(x, y) \in \mathbb{R}^2\colon\ \left(x - \frac{1}{2}\right)^2 + y^2 < \frac{1}{4}\right\}.$$

Then $Q = Q_* = \{(1, 0)\}$ and $L(Q) = 0$ for any f. Consider the particular choice

$$f(x, y) = \alpha \arcsin|y|, \quad (x, y) \in \partial\Omega\ (\alpha > 0).$$

Thus, for $\alpha < 1$, this function is of type A, since

$$\begin{aligned}\left|f(x_1, y_1) - f(x_2, y_2)\right| &= \alpha\left|\arcsin|y_1| - \arcsin|y_2|\right| \\ &\leqslant \left||y_1| - |y_2|\right| \\ &\leqslant \left|(x_1, y_1) - (x_2, y_2)\right|,\end{aligned}$$

for small y_1 and y_2 (because $\arcsin'(0) = 1$).

For $\alpha = 1$, f is of type B. Indeed, it is not of type A since $\arcsin|y| > |(x, y)|$ for $(x, y) \in \partial\Omega$, $y \neq 0$. Moreover, for any $\varepsilon > 0$, $\arcsin|y| \leqslant (1 + \varepsilon)|y| \leqslant (1 + \varepsilon)|(x, y)|$, for small y. Furthermore, it can be decomposed as $f = f_A + f_0$, $f_A = f - f^2$, $f_0 = f^2$. The derivative of $\arcsin t - \arcsin^2 t$ is

$$\frac{1 - 2\arcsin t}{\sqrt{1 - t^2}}$$

and, since it does not exceed one near zero, f_A is of type A. Conditions (9) and (10) hold with $\gamma = \frac{1}{2}$. The second one is clear, and the first follows from a more general reasoning (cf. the next example), although it can also be checked directly. Thus, Theorem 3.2(ii) is applicable in this case.

For $\alpha > 1$, f is of type C (we can take $\varepsilon = \alpha - 1$). But (cf. the next example)

$$\gamma(1, 0) = \overline{\gamma}(1, 0) = \frac{1}{2}.$$

Thus, (11) holds if $p(1, 0) < 3$, and (21) is true for $p(1, 0) > 3$.

In conclusion, we have:

- $S_L \neq \emptyset$ for $\alpha \leqslant 1$;
- $S \neq \emptyset$ but $S_L = \emptyset$ for $p(1, 0) < 3$ and $\alpha > 1$;
- S is empty when $p(1, 0) > 3$ and $\alpha > 1$.

The proof of the last statement follows closely that of Theorem 4.1(ii), with the inequality

$$\begin{aligned}\big|f_2(x,y)\big| &\geqslant \big|f(x,y)\big| - \big|f_1(x,y) - f_1(1,0)\big| \\ &\geqslant \alpha\big|(x-1,y)\big| - \big|(x-1,y)\big| \\ &= (\alpha-1)\big|(x,y)-(1,0)\big|\end{aligned}$$

replacing (24).

Example 5.2. Now we consider a more general example. Assume that Q consists of only one point z. Let H be the tangent hyperspace to $\partial\Omega$ at the point z. We may assume, w.l.o.g., that $z = 0$, $H = \mathbb{R}^{N-1}_*$ and $\Omega \subset \mathbb{R}^N_+$. Each element of $\mathbb{R}^N$ may be identified with a vector (x, y), where $x \in \mathbb{R}^{N-1}_*$ and $y \in \mathbb{R}$. Assume that ∂D and $\partial\Omega$ are C^2-smooth. Then the sets ∂D and $\partial\Omega$ near z are graphs of certain C^2-smooth functions $\phi(x)$ and $\psi(x)$ defined for small $x \in \mathbb{R}^{N-1}$, with $\phi(x) \geqslant \psi(x)$. Assume that the contact of ∂D and $\partial\Omega$ is *simple* in the sense that the quadratic form $(\phi-\psi)''(0)$ is non-degenerate (and thus positive-definite). Then

$$\gamma(z) = \overline{\gamma}(z) = \frac{1}{2}.$$

Indeed, for any $w = (x, \psi(x)) \in \partial\Omega$, we have

$$\begin{aligned}|w|^2 &\leqslant c|x|^2 \leqslant c\big(\phi(x)-\psi(x)\big) \\ &\leqslant c\big|w - d(w)\big| \leqslant c\big(\phi(x)-\psi(x)\big) \\ &\leqslant c|x|^2 \leqslant c|w|^2.\end{aligned}$$

Therefore, by Theorem 3.2(i) and (iii), S is always non-empty for $p(z) < N+1$ (f is of type A, B or C since $L(Q) = 0$). By Theorem 4.1(ii), S is empty for some f of type C when $p(z) > N+1$.

Example 5.3. We remain in the framework of the previous example, but now the sets ∂D and $\partial\Omega$, and the functions $\phi(x)$ and $\psi(x)$ may be only C^1-smooth (and we do not know if the contact is simple). Assume that there exist $\alpha > 1$ and positive constants C_1 and C_2 such that

$$C_1|x|^\alpha \leqslant \phi(x) - \psi(x) \leqslant C_2|x|^\alpha. \tag{28}$$

Then

$$\gamma(z) = \overline{\gamma}(z) = \frac{1}{\alpha}.$$

In fact, for any $w = (x, \psi(x)) \in \partial\Omega$, we have

$$\begin{aligned}|w|^\alpha &\leqslant c|x|^\alpha \leqslant c\big(\phi(x)-\psi(x)\big) \\ &\leqslant c\big|w - d(w)\big| \leqslant c\big(\phi(x)-\psi(x)\big) \\ &\leqslant c|x|^\alpha \leqslant c|w|^\alpha.\end{aligned}$$

Thus, the value of $p(z)$ determines whether Theorem 3.2(iii) or Theorem 4.1(ii) is applicable.

Example 5.4. A limiting case of the previous example arises when $\alpha \to 1$ in (28). More precisely, we assume

$$|x| \leqslant c\big(\phi(x) - \psi(x)\big). \tag{29}$$

In this case, ∂D and $\partial\Omega$ can only be topological manifolds (for $\phi(x) - \psi(x)$ cannot be differentiable at zero). Assume, for simplicity, that $N = 2$. Since the domains Ω and D are convex, the functions $\phi(x)$ and $\psi(x)$ are convex, and thus are locally Lipschitz, and have locally bounded left and right derivatives (see e.g. [6, Chapter 1]), so the angle

$$\big(\big(x,\psi(x)\big), \big(x,\phi(x)\big), d\big(x,\psi(x)\big)\big)$$

does not go to zero (for small x). This implies (cf. the proof of Theorem 4.1(ii))

$$\phi(x) - \psi(x) \leqslant c\big|\big(x,\psi(x)\big) - d\big(x,\psi(x)\big)\big|.$$

We fix some small neighbourhood W of z, and put $W_1 = Q = \{z\}$, $W_2 = (\partial\Omega \cap W)\backslash Q$. Then $L(W_1) = 0$ and

$$\begin{aligned}|w| &\leqslant c|x| \leqslant c\big(\phi(x)-\psi(x)\big) \\ &\leqslant c\big|w - d(w)\big| \\ &\leqslant c|w - v|,\end{aligned}$$

for all $w = (x, \psi(x)) \in \partial\Omega \cap W$ and $v \in \partial D \cap W$. We can apply Theorem 3.1 to conclude that $S_L \neq \emptyset$, for any f and p.

Example 5.5. A related smooth example, which can also be considered a limiting case of Example 5.3, is the following:

$$\phi(x) - \psi(x) = \frac{|x|}{\ln(|x|^{\zeta})}, \quad \zeta < 0.$$

For all $\gamma < 1$ and $w = (x, \psi(x))$, we have

$$\begin{aligned} |w| &\leqslant c|x| \leqslant c\frac{|x|^{\gamma}}{(\ln(|x|^{\zeta}))^{\gamma}} \\ &= c\big(\phi(x) - \psi(x)\big)^{\gamma} \\ &\leqslant c\big|w - d(w)\big|^{\gamma}. \end{aligned}$$

Hence, $\gamma(z) = 1$, and (11) is always valid. Therefore $S \neq \emptyset$, for every f and p.

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